Mathematik II

2011-09-19

1) erste Gerade: $\begin{pmatrix}1\\2\end{pmatrix} + \lambda \begin{pmatrix}3\\3\end{pmatrix}$

zweite " : $\begin{pmatrix}4\\3\end{pmatrix} + \mu \begin{pmatrix}-2\\3\end{pmatrix}$

Schnittpunkt? $\begin{pmatrix}1\\2\end{pmatrix} + \lambda \begin{pmatrix}3\\3\end{pmatrix} \overset{!}{=} \begin{pmatrix}4\\3\end{pmatrix} + \mu \begin{pmatrix}-2\\3\end{pmatrix}$

$$\Longleftrightarrow \begin{cases} 3\lambda + 2\mu = 3 & \text{I} \\ 3\lambda - 3\mu = 1 & \text{II} \end{cases}$$

$$\Longleftrightarrow \begin{cases} 5\mu = 2 & \text{I}-\text{II} \\ 3\lambda = 1 + 3\mu & \text{II} \end{cases}$$

$$\Longleftrightarrow \mu = \frac{2}{5} \wedge \lambda = \frac{1}{3}\left(1 + \frac{6}{5}\right) = \frac{11}{15}$$

Also genau ein Schnittpunkt:

$$\begin{pmatrix}4\\3\end{pmatrix} + \frac{2}{5}\begin{pmatrix}-2\\3\end{pmatrix} \left(\left(= \begin{pmatrix}4-4/5\\3+6/5\end{pmatrix} = \begin{pmatrix}3\,1/5\\4\,1/5\end{pmatrix}\right)\right)$$

2) $$\begin{vmatrix} 1 & 1 & 0 & 1 \\ 3 & 0 & -5 & 2 \\ 0 & 0 & 4 & 1 \\ 2 & 2 & 1 & 2 \end{vmatrix} = \begin{array}{c} \begin{matrix} + & - & + & - \end{matrix} \\ \begin{vmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & -5 & 2 \\ 0 & 0 & 4 & 1 \\ 0 & 2 & 1 & 0 \end{vmatrix} \end{array}$$

Entwickeln nach 1. Zeile

$$= -\left|\begin{matrix}3 & -5 & 2\\ 0 & 4 & 1\\ 0 & 1 & 0\end{matrix}\right| = -3\cdot\left|\begin{matrix}4 & 1\\ 1 & 0\end{matrix}\right| = 3.$$

Entwickeln nach 1. Spalte

3) Ansatz: $y = e^{\lambda x}$

$$\Rightarrow \lambda^2 + 3 = 0 \Rightarrow \lambda = \pm\sqrt{3}\,i$$

Also allgemeine Lösung:

$$y = A e^{\sqrt{3}\,ix} + B e^{-\sqrt{3}\,ix}$$

Anfangsbedingung:

$$\begin{cases} 5 \overset{!}{=} y(0) = A + B \\ 0 \overset{!}{=} y'(0) = \sqrt{3}\,i(A-B) \end{cases}$$

Also $A = B = 5/2$.

4) $$a_0 = \frac{2}{5}\int_0^5 f(t)\,dt = \frac{2}{5}\int_4^5 3\,dt = \frac{6}{5}$$

$$a_2 = \frac{2}{5}\int_0^5 f(t)\cos\left(2\pi 2\frac{t}{5}\right)dt$$

$$= \frac{2}{5}\int_4^5 3\cos\left(2\pi\cdot 2\,\frac{t}{5}\right)dt$$

$$= \frac{\cancel{6}^{\,3}}{5}\left[\frac{\cancel{5}}{\cancel{4}_{2}\pi}\sin\left(\frac{4\pi}{5}t\right)\right]_4^5$$

$$= \frac{3}{2\pi}\left(\underbrace{\sin\left(\frac{4\pi\cancel{5}}{\cancel{5}}\right)}_{0} - \sin\left(\frac{4\pi 4}{5}\right)\right)$$

$$= -\frac{3}{2\pi}\sin\left(\frac{16}{5}\pi\right)$$

5) $f(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$

$$f'(x) = -\frac{1}{2}x^{-3/2}$$

$$f''(x) = \frac{3}{4}x^{-5/2}$$

Schmiegeparabel bei $x_0 = 4$:

$$y = \frac{1}{\sqrt{4}} - \frac{1}{2}4^{-3/2}(x-4) + \frac{3}{4}4^{-5/2}\frac{(x-4)^2}{2}$$

$$= \frac{1}{2} - \frac{1}{16}(x-4) + \frac{3}{256}(x-4)^2$$

Also Schätzung für $x = 4{,}01$:

$$y = \frac{1}{2} - \frac{1}{16} \cdot \frac{1}{100} + \frac{3}{256} \cdot \frac{1}{10000}$$

6) $\frac{\partial f}{\partial x} = -2x + 6y + 8$

$\frac{\partial f}{\partial y} = 6x - 2y - 16$

Notwendig für lok. Extremum: beides $= 0$:

$$\left\{ \begin{array}{l} -2x + 6y = -8 \\ 6x - 2y = 16 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} -x + 3y = -4 \\ 3x - y = 8 \end{array} \right.$$

Lösung z.B. mit Cramer:

$$x = \frac{\begin{vmatrix} -4 & 3 \\ 8 & -1 \end{vmatrix}}{\begin{vmatrix} -1 & 3 \\ 3 & -1 \end{vmatrix}} = \frac{-20}{-8} = 5/2$$

$$y = \frac{\begin{vmatrix} -1 & -4 \\ 3 & 8 \end{vmatrix}}{\begin{vmatrix} -1 & 3 \\ 3 & -1 \end{vmatrix}} = \frac{4}{-8} = -\frac{1}{2}$$

Also lok. Extr. allenfalls

an $(x|y) = \left(\frac{5}{2} \middle| -\frac{1}{2}\right)$.

Prüfe mit Hesse:

$$\frac{\partial^2 f}{\partial x^2} = -2, \quad \frac{\partial^2 f}{\partial x \partial y} = 6, \quad \frac{\partial^2 f}{\partial y^2} = -2$$

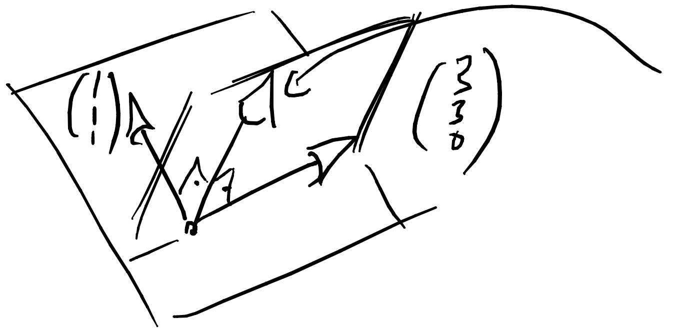
$$\text{Hesse-Matrix} = \begin{pmatrix} -2 & 6 \\ 6 & -2 \end{pmatrix}$$

$$\det(\text{''}) = 4 - 36 < 0$$

Kann immer nur Sattel sein!

$\Rightarrow$ kein lok. Extremum

7) Erste Ebene in Punkt-Richtungs-Form: $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.



z.B. $\begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}$

Andere Ebene z.B.:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}$$

8) $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 3 & 0 & 3 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}$

$+$ $\cdot 2$

Also nur zwei linear unabhängige Spalten

$\Longrightarrow$ Rang $= 2$.

Rang + Defekt $= 4$

$\Longrightarrow$ Defekt $= 2$.

9) $y' = x^3 y \Longleftrightarrow \frac{y'}{y} = x^3$

$$\Longleftrightarrow \underbrace{\int_{-7}^{y_1} \frac{dy}{y}}_{[\ln|y|]_{-7}^{y_1}} = \underbrace{\int_{5}^{x_1} x^3 dx}_{[\frac{x^4}{4}]_5^{x_1}}$$

$$\Longleftrightarrow \ln|y_1| = \ln|-7| + \frac{x_1^4}{4} - \frac{5^4}{4}$$

$$\Longleftrightarrow \quad y_1 = -7 \exp\left(\frac{x_1^4}{4} - \frac{5^4}{4}\right)$$

wg. Anfangsbedingung

10) $y' - 2y \overset{!}{=} e^{2x}$

Ansatz: $y = Ae^{2x}$

$$\Rightarrow \underbrace{A\,2e^{2x} - 2Ae^{2x}}_{0} \overset{!}{=} e^{2x} \;\; ↯$$

Neuer Ansatz: $y = Axe^{2x}$

$$\Rightarrow Ae^{2x} + \cancel{Ax2e^{2x}} - \cancel{2Axe^{2x}} \overset{!}{=} e^{2x}$$

Also wähle $A = 1$.

11) $$\frac{s+1}{(s-2)s^2} \overset{!}{=} \frac{A}{s-2} + \frac{B}{s} + \frac{C}{s^2}$$

Partialbruchzerlegung

$$= \frac{As^2 + Bs(s-2) + C(s-2)}{(s-2)s^2}$$

$$\Rightarrow A = \frac{3}{4} \wedge C = -\frac{1}{2} \wedge B = -\frac{3}{4}$$

s=2 einsetzen

s=0 einsetzen

z.B. s=1 einsetzen

Also ... $= \frac{3}{4}\frac{1}{s-2} - \frac{3}{4}\frac{1}{s} - \frac{1}{2}\frac{1}{s^2}$

$$\begin{array}{ccc} \frac{1}{s-2} & \frac{1}{s} & \frac{1}{s^2} \\ \Big\uparrow \mathcal{L} & \Big\uparrow & \Big\uparrow \\ e^{2t} & 1 & t \end{array}$$

$\Rightarrow$ Original function

$$= \frac{3}{4}e^{2t} - \frac{3}{4} - \frac{t}{2}$$

12) Integral $= \int_0^3 \int_0^{\pi/2} \int_0^{2\pi} \underbrace{z}_{r\cos\vartheta} r^2 \sin\vartheta \, d\varphi \, d\vartheta \, dr$

$$= \int_0^3 \underbrace{\left(\int_0^{\pi/2} \underbrace{\left(\int_0^{2\pi} r^3 \frac{1}{2}\sin(2\vartheta)\, d\varphi \right)}_{\frac{r^3}{\cancel{2}}\sin(2\vartheta)\cdot \cancel{2}\pi} d\vartheta \right)}_{\pi r^3 \left[-\frac{\cos(2\vartheta)}{2}\right]_0^{\pi/2} = \pi r^3 \cdot \frac{1+1}{2}} dr$$

$$= \pi \int_0^3 r^3 \, dr = \pi \left[\frac{r^4}{4}\right]_0^3 = \pi \frac{3^4}{4}.$$