Mathematik II

2013-09-25

1) $$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\left| \; \% \; \right| = 3+0-8-0-0-0$$

$$= -5$$

$$x = \frac{1}{-5} \begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 4 & 3 \end{vmatrix}$$

$$= \frac{3+0-8+3-0-0}{-5}$$

$$= 2/5$$

2) Eigenwerte:

$$0 = \begin{vmatrix} 3-\lambda & 1 & 2 \\ 0 & -1-\lambda & 3 \\ 0 & 0 & 4-\lambda \end{vmatrix}$$

$$= (3-\lambda)(-1-\lambda)(4-\lambda)+0+0-0-0-0$$

Also $\lambda = 3 \vee \lambda = -1 \vee \lambda = 4$.

Wir nehmen z.B. einen Eigenvektor zu $\lambda = 3$ und einen zu $\lambda = -1$.

z.B. EV zu EW 3:

$$\underbrace{\begin{pmatrix} 3-3 & 1 & 2 \\ 0 & -1-3 & 3 \\ 0 & 0 & 1 \end{pmatrix}}_{\begin{pmatrix} 0 & 1 & 2 \\ 0 & -4 & 3 \\ 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Nehme z.B. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

z.B. EV zu EW -1:

$$\begin{pmatrix} 4 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Nehme z.B. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}$

3) $y' + 3y = \sin(x)$

Ansatz: $y(x) = A\sin(x) + B\cos(x)$

$$\Rightarrow A\cos(x) - B\sin(x)$$
$$+ 3A\sin(x) + 3B\cos(x) = \sin(x)$$

$$\Rightarrow \left\{ \begin{matrix} A + 3B = 0 \\ 3A - B = 1 \end{matrix} \right\} \Rightarrow \begin{cases} A = \dfrac{\begin{vmatrix} 0 & 3 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix}} = \dfrac{-3}{-10} \\[2ex] B = \dfrac{\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix}}{-10} = -\dfrac{1}{10} \end{cases}$$

4) $f(x) = (\ln(x))^2$

$$f'(x) = 2\ln(x) \cdot \frac{1}{x}$$

$$f''(x) = 2\,\frac{1}{x} \cdot \frac{1}{x} + 2\ln(x) \cdot \frac{-1}{x^2}$$

$$f(e) = 1, \quad f'(e) = \frac{2}{e}$$

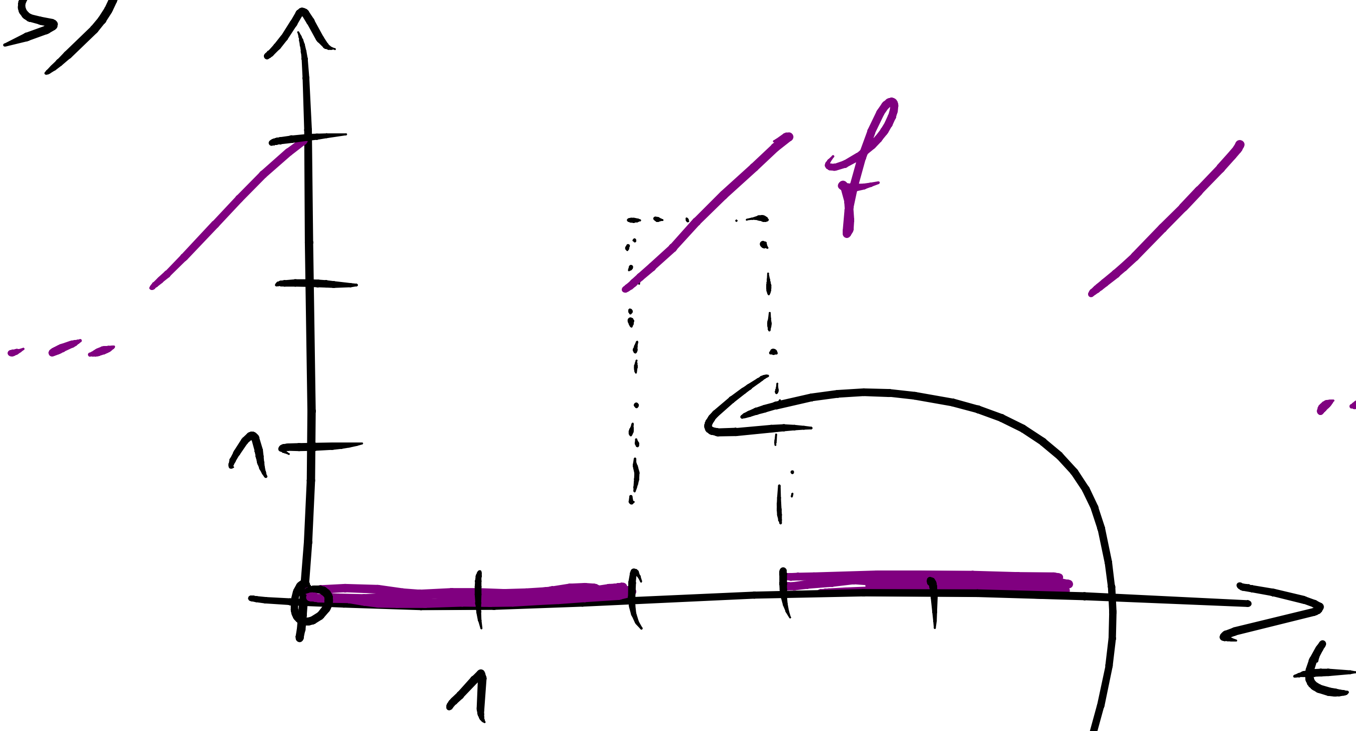
$$f''(e) = \frac{2}{e^2} + 2 \cdot \frac{-1}{e^2} = 0$$

Also:

$$y \approx 1 + \frac{2}{e}(x-e)$$

((kein x^2!))

5)



$$c_0 = \frac{2\frac{1}{2}}{3} = \frac{5}{6}$$

((oder per Integral))

$$c_2 = \frac{1}{3} \int_0^3 e^{-2\pi i 2 \frac{t}{3}} f(t)\, dt$$

$$= \frac{1}{3} \int_2^3 e^{-\frac{4}{3}\pi i t}\, t\, dt$$

$$\frac{e^{-\frac{4}{3}\pi i t}}{-\frac{4}{3}\pi i} \qquad 1 \qquad 3$$

$$= \frac{1}{\cancel{3}}\left[\frac{e^{-\frac{4}{3}\pi i t}}{-\frac{4}{\cancel{3}}\pi i}\, t\right]_2^3 - \frac{1}{\cancel{3}}\int_2 \frac{e^{-\frac{4}{3}\pi i t}}{-\frac{4}{\cancel{3}}\pi i}\, dt$$

$$= \frac{i}{4\pi}\Big(3\underbrace{e^{-4\pi i}}_{1} - 2e^{-\frac{8}{3}\pi i}\Big)$$

$$- \frac{\cancel{i}}{4\pi}\left[\frac{e^{-\frac{4}{3}\pi i t}}{-\frac{4}{3}\pi \cancel{i}}\right]_2^3$$

$$\Big(\Big(= \frac{3i}{4\pi} + \frac{3}{16\pi^2} - e^{-\frac{8}{3}\pi i}\Big(\frac{i}{2\pi}+\frac{3}{16\pi^2}\Big)\Big)\Big)$$

6) $f(x,y) = (4x-x^2)e^{y^2-2y}$

$$\frac{\partial f}{\partial x} = (4-2x)e^{\dots}$$

$$\frac{\partial f}{\partial y} = (4x-x^2)\, e^{\dots}\cdot(2y-2)$$

$$\frac{\partial f}{\partial x}=0 \iff x=2$$

unmöglich

$$\frac{\partial f}{\partial y}=0 \iff x=0 \vee x=4 \vee y=1$$

$$\frac{\partial f}{\partial x}=0 \wedge \frac{\partial f}{\partial y}=0 \iff x=2 \wedge y=1$$

$$\frac{\partial^2 f}{\partial x^2} = -2e^{\cdots} \longrightarrow -2e^{-1}$$

$$\frac{\partial^2 f}{\partial y^2} = (4x-x^2)e^{\cdots}\cdot\left((2y-2)^2+2\right) \rightarrow 4e^{-1}\cdot 2$$

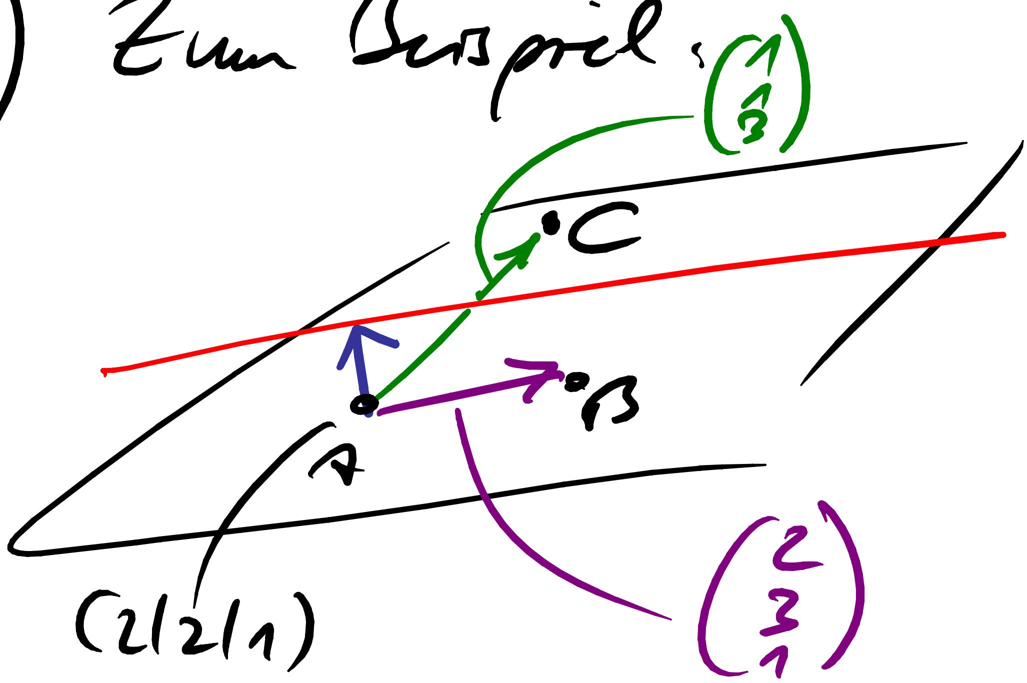
$$\frac{\partial^2 f}{\partial x \partial y} = (4-2x)e^{\cdots}(2y-2) \rightarrow 0$$

$\Longrightarrow$ Hesse-Matrix an $x=2, y=1$ ist

$$\begin{pmatrix} -2e^{-1} & 0 \\ 0 & 8e^{-1} \end{pmatrix}.$$

$\det(\cdot/\cdot) < 0$, also Sattelpunkt. Es gibt kein lok. Max/Min.

7) Zum Beispiel:



Also Gerade z.B.

$$\begin{pmatrix}2\\2\\1\end{pmatrix} + \underbrace{\boxed{\begin{pmatrix}2\\3\\1\end{pmatrix}\times\begin{pmatrix}1\\1\\3\end{pmatrix}}}_{\begin{pmatrix}8\\-5\\-1\end{pmatrix}} + \lambda \boxed{\begin{pmatrix}2\\3\\1\end{pmatrix}} = \begin{pmatrix}10\\-3\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\1\end{pmatrix}$$

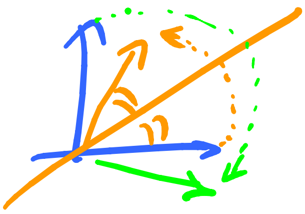
8) $\begin{pmatrix}\frac{1}{2} & a\\ \frac{\sqrt{3}}{2} & b\end{pmatrix}$ Spiegelung?

Brauchen:

- $-1 = \det(\ldots) = \frac{1}{2}b - \frac{\sqrt{3}}{2}a$
- $\begin{pmatrix}a\\b\end{pmatrix}$ hat Länge 1, d.h. $a^2+b^2=1$

Wähle $a=\frac{\sqrt{3}}{2}$ und $b=-\frac{1}{2}$.

((Oder rein grafisch:



1. Spalte
2. Spalte))

g) $\frac{y}{y'} = x \implies \frac{y'}{y} = \frac{1}{x}$

$$\implies \frac{dy}{y} = \frac{dx}{x}$$

Also $\underbrace{\int_2^{y_1} \frac{dy}{y}}_{[\ln|y|]_2^{y_1}} = \underbrace{\int_5^{x_1} \frac{dx}{x}}_{[\ln|x|]_5^{x_1}}$

$$\implies \ln|y_1| - \ln(2) = \ln|x_1| - \ln(5)$$

$$\implies |y_1| = |x_1| \cdot \frac{2}{5} \implies y_1 = \pm \frac{2}{5} x_1$$

Muss „+" sein, damit $y(x)$ stetig ist (wegen $y(5)=2$)

10) $y'' + 6y' + 10y = 42 \quad (*)$

spez. Lösung: $y = 42$

allg. Lösung für $\ldots = 0$:

$\lambda^2 + 6\lambda + 10 = 0$

$\Rightarrow \lambda = -3 \pm \sqrt{9-10} = -3 \pm i$

allg. Lösung für $(*)$:

$$y = A e^{(-3+i)x} + B e^{(-3-i)x} + 42$$

$$A e^{(-3+i)x} \to 0, \quad B e^{(-3-i)x} \to 0$$

Alle Lösungen gehen $\to 42$, keine $\to 0$.

11) $\frac{1}{s^3+5s} = \frac{1}{s(s^2+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+5}$

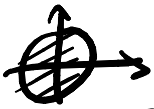
$= \frac{A(s^2+5)+(Bs+C)s}{s(s^2+5)} = \frac{s^2(A+B)+sC+5A}{s(s^2+5)}$

$\Rightarrow \begin{cases} A+B=0 \\ C=0 \\ 5A=1 \end{cases} \Rightarrow \begin{cases} A = 1/5 \\ B = -1/5 \\ C = 0 \end{cases}$

$\Rightarrow \frac{1}{s^3+5s} = \frac{1}{5}\cdot\frac{1}{s} - \frac{1}{5}\frac{s}{s^2+5}$

$\Rightarrow y(t) = \frac{1}{5} - \frac{1}{5}\cos(\sqrt{5}t)$

12) $$V = \iint (1+\cos(x^2+y^2))\,dx\,dy$$



$$= \int_0^5 \Big(\underbrace{\int_0^{2\pi} (1+\cos(r^2))\,d\varphi}_{2\pi(1+\cos(r^2))}\Big)\, r\,dr$$

$$= 2\pi \int_0^5 (1+\cos(r^2))\, r\,dr$$

$$= \underbrace{2\pi \int_0^5 r\,dr}_{\cancel{2}\pi \frac{5^2}{\cancel{2}}} + 2\pi \underbrace{\int_0^5 \cos(r^2)\, r\,dr}_{}$$

$u = r^2$; $du = 2r\,dr$

$$\int_0^{25} \cos(u)\,\frac{du}{2}$$

$$= \frac{1}{2}\big[\sin(u)\big]_0^{25}$$

$$= 25\pi + \cancel{2}\pi \frac{1}{\cancel{2}} \sin(25)$$