Mathematik 2
2023-07-14
Musterlösungen

1)
$$\begin{pmatrix} 3 & 8 & 0 \\ 2 & 4 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longleftrightarrow \begin{cases} 3x + 8y = 0 & \mathrm{I} \\ 2x + 4y + 6z = 0 & \mathrm{II} \end{cases}$$

$$2 \cdot \mathrm{II} - \mathrm{I} : \quad x + 12z = 0$$

$$\Longleftrightarrow x \text{ frei wählbar}, \; y = -\frac{3}{8}x, \; z = -\frac{1}{12}x$$

Also Kern = Gerade: $\lambda \begin{pmatrix} 1 \\ -3/8 \\ -1/12 \end{pmatrix}$.

2)

Entwicklung nach 4. Spalte

$$z = \frac{\begin{vmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 3 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 1 & 0 & -1 \end{vmatrix}} = \frac{\cancel{(-1)} \cdot \begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix}}{\cancel{(-1)} \cdot \begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -5 \end{vmatrix}}$$

$$= \frac{0 + 0 + 0 - 0 - 1 - (-2)}{0 + 0 + 0 - 0 - (-1) - 5} = -\frac{1}{4}$$

3) • Allg. Lsg. der hom. Form:

$$\ddot{x} + 6\dot{x} + 11x \stackrel{!}{=} 0$$

Ansatz: $x(t) = e^{\lambda t}$

$$\Rightarrow \lambda^2 + 6\lambda + 11 = 0$$

$$\Rightarrow \lambda = -3 \pm \sqrt{9-11}$$

$$= -3 \pm \sqrt{2}\, i$$

Also allg. Lsg. der hom. Form:

$$x(t) = A e^{(-3+\sqrt{2} i)t} + B e^{(-3-\sqrt{2} i)t}$$

• Eine spez. Lsg. der inh. Form:

$$\ddot{x} + 6\dot{x} + 11x \stackrel{!}{=} 23$$

Ansatz: $x(t) = C$

$$\Rightarrow C = 23/11$$

• Allg. Lösung der ursprünglichen DGL:

$$x(t) = A e^{(-3+\sqrt{2} i)t} + B e^{\cdots} + \frac{23}{11}$$

4) Trennung der Variablen: $\frac{dy}{y^3} = \frac{dx}{x^2}$

Integration: $\int_5^{y_1} \frac{dy}{y^3} = \int_3^{x_1} \frac{dx}{x^2}$

$$\Rightarrow \left[-\frac{1}{2}y^{-2}\right]_5^{y_1} = \left[-x^{-1}\right]_3^{x_1}$$

$$\Rightarrow -\frac{1}{2\cdot y_1^2} + \frac{1}{2\cdot 25} = -\frac{1}{x_1} + \frac{1}{3}$$

$$\Rightarrow y_1 = \pm \frac{1}{\sqrt{2}\sqrt{\frac{1}{x_1} \underbrace{-\frac{1}{3} + \frac{1}{50}}_{-47/150}}}.$$

+ wegen Anfangswert

5. Gerade Funktion $\Rightarrow b_{\dots} = 0$

$$a_5 = \frac{\cancel{2}^{1}}{\cancel{10}_5}\int_2^8 \cos(\underbrace{2\pi\cdot 5t/10}_{\pi\cdot t})\cdot 3\,dt$$

$$= \frac{1}{5}\cdot 3\cdot\left[\frac{\sin(\pi t)}{\pi}\right]_2^8$$

$$= 3\cdot\frac{\sin(8\pi) - \sin(2\pi)}{5\pi} = 0$$

6. $$\frac{s+2}{s^2-9} = \frac{A}{s-3} + \frac{B}{s+3} = \frac{A(s+3)+B(s-3)}{(s-3)(s+3)}$$

$$\Leftrightarrow \left\{\begin{array}{ll} s^1 & 1 = A+B \\ s^0 & 2 = 3A-3B \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} A = \frac{\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 3 & -3 \end{vmatrix}} = \frac{-5}{-6} = \frac{5}{6} \\ B = 1 - \frac{5}{6} = \frac{1}{6} \end{array}\right.$$

Also $y(t) = \frac{5}{6}e^{3t} + \frac{1}{6}e^{-3t}$.

7. Richtungsvektoren der beiden Geraden: $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ und $\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$.

Der gesuchte Richtungsvektor ist $\perp$ zu beiden.

Nehme z.B. $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \times \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -8 \\ 16 \\ -6 \end{pmatrix}$.

8. Ansatz: $y(x) = A\sin(2x) + B\cos(2x)$

Einsetzen:

$$-8A\cos(2x) + 8B\sin(2x) - 9A\sin(2x) - 9B\cos(2x) \overset{!}{=} \sin(2x)$$

$$\Longleftrightarrow \begin{cases} (\sin)\ -9A + 8B = 1 \\ (\cos)\ -8A - 9B = 0 \end{cases}$$

$$\Longleftrightarrow A = \frac{\begin{vmatrix} 1 & 8 \\ 0 & -9 \end{vmatrix}}{\begin{vmatrix} -9 & 8 \\ -8 & -9 \end{vmatrix}} = \frac{-9}{145}, \quad B = \frac{\begin{vmatrix} -9 & 1 \\ -8 & 0 \end{vmatrix}}{\begin{vmatrix} -9 & 8 \\ -8 & -9 \end{vmatrix}} = \frac{8}{145}.$$

9. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

und $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} \overset{!}{=} 2 \cdot \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ $\Longrightarrow 3a = 6$ $\Longrightarrow 3c = 0$

Also $a = 2$, $c = 0$.

$$\Longrightarrow \begin{pmatrix} 2 & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \begin{cases} 10 + 6b = 0 \\ 6d = 0 \end{cases}.$$

Also $b = -\frac{5}{3}$, $d = 0$.

10. an $x_0 = 0$:

$$f(x) = \sin(\sin(x)) \qquad 0$$

$$f'(x) = \cos(\sin(x)) \cdot \cos(x) \qquad 1$$

$$f''(x) = -\sin(\sin(x)) \cdot (\cos(x))^2 + \cos(\sin(x)) \cdot (-\sin(x)) \qquad 0$$

Also $\sin(\sin(0{,}01)) \approx \underbrace{0 + 1 \cdot 0{,}01 + 0 \cdot \frac{0{,}01^2}{2}}_{0{,}01}$

11. $6 - \sqrt{x^2+y^2} \overset{!}{=} 1$

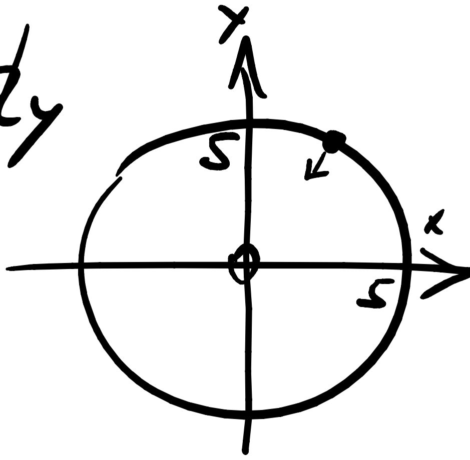
$\Longleftrightarrow \sqrt{x^2+y^2} = 5$

Kreislinie mit Radius 5 um Ursprung!

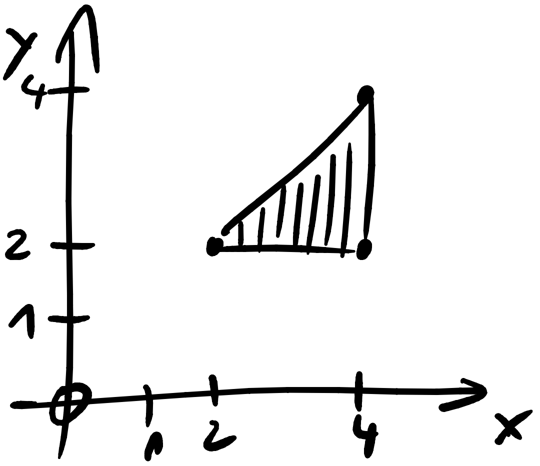
Gradient: $\frac{\partial f}{\partial x} = -\frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cdot \cancel{2}x$

$$\frac{\partial f}{\partial y} = -\frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cdot \cancel{2}y$$

Also $\operatorname{grad} f(3,4) = \begin{pmatrix} -3/5 \\ -4/5 \end{pmatrix}$.



12.



Zum Beispiel längs y schneiden

$$V = \int_2^4 \left(\underbrace{\int_2^x xy \, dy}_{\left[x\frac{y^2}{2} \right]_{y=2}^{y=x}} \right) dx$$

$$x \frac{x^2}{2} - x \cdot \frac{4}{2}$$

$$= \int_2^4 \left(\frac{x^3}{2} - 2x \right) dx$$

$$= \left[\frac{x^4}{8} - x^2 \right]_2^4$$

$$= \frac{\cancel{16}^{2} \cdot 16}{\cancel{8}_{1}} - 16 - \cancel{\frac{16}{8}}_{2} + 4 = 18.$$